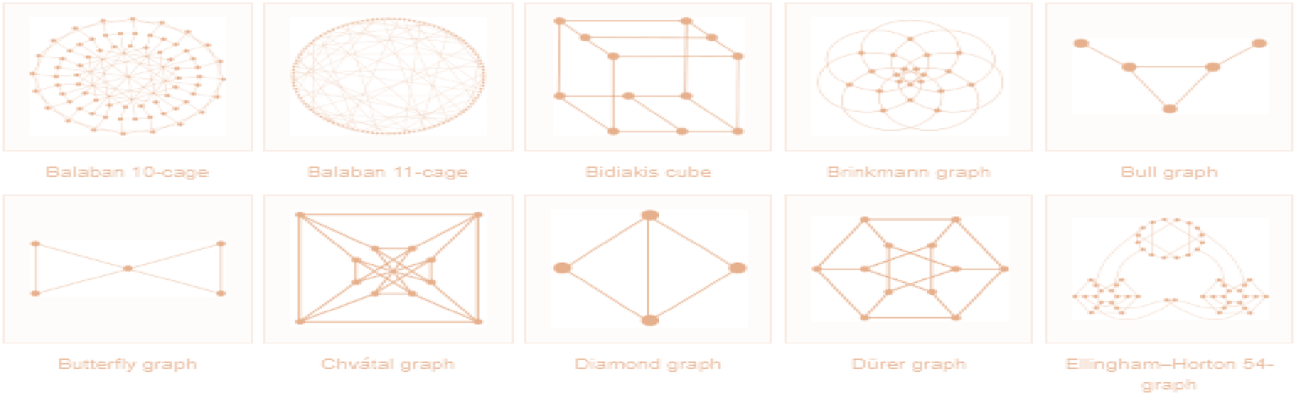
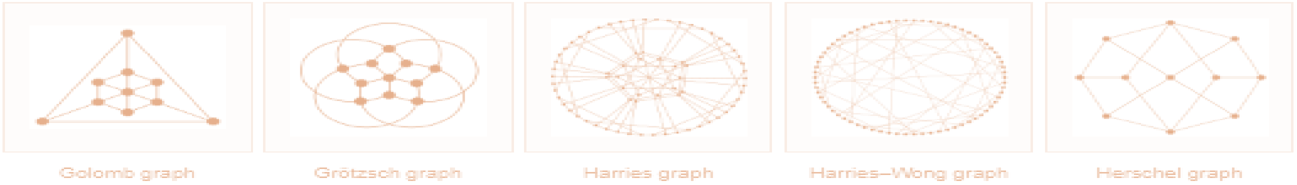
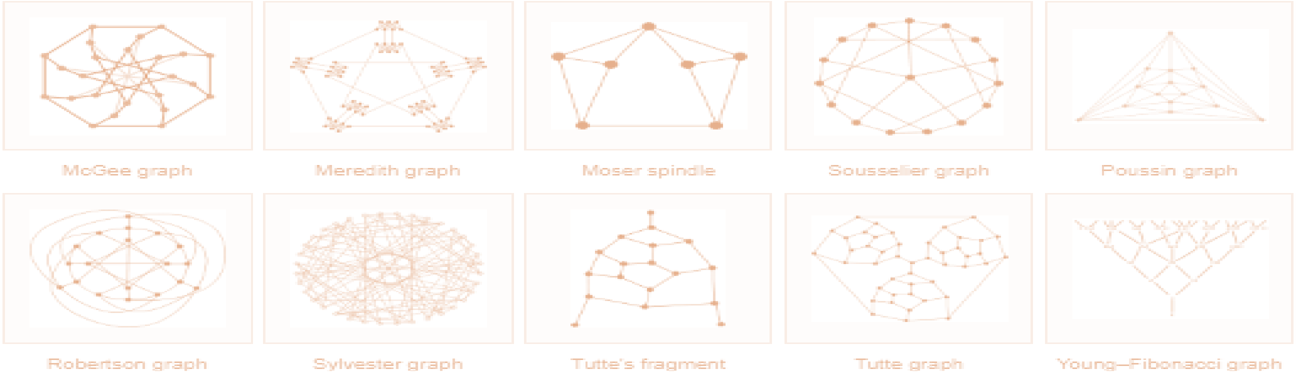


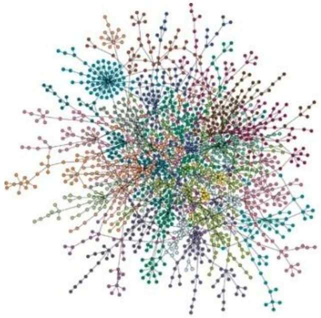
Graph Theory

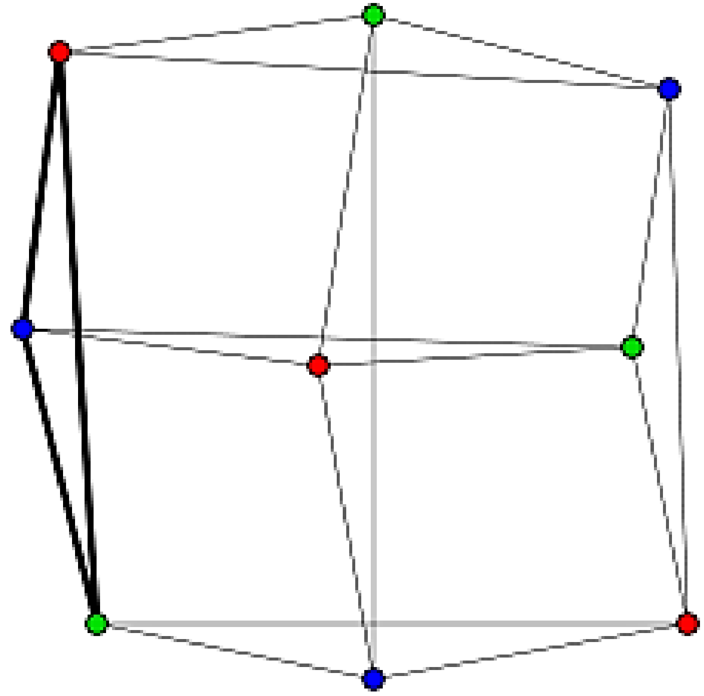


Perfect Graphs



Planarity





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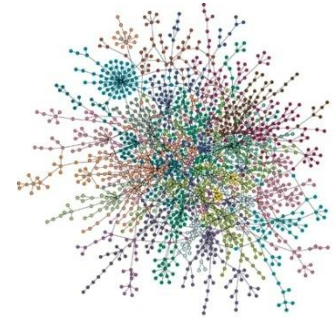


Perfect Graphs

- A **perfect graph** is a graph in which the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph (clique number).

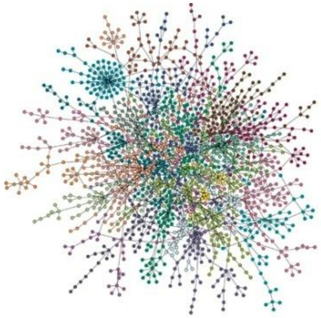
- An arbitrary graph G is perfect if and only if we have:

$$\forall\, S \subseteq V(G)\big(\chi\,(G[S]) = \omega\,(G[S])\big)$$

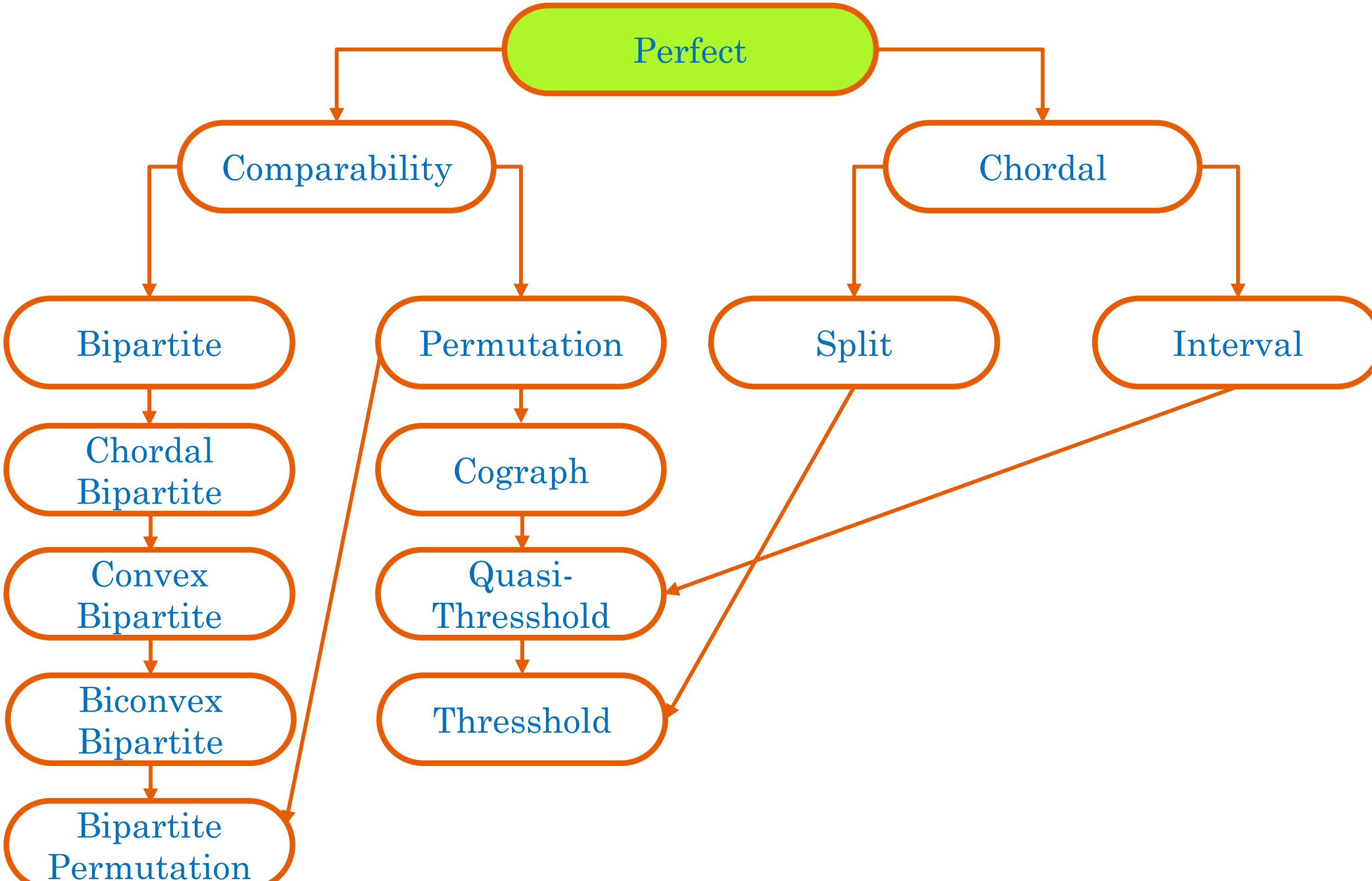
- **Theorem 1 (Perfect Graph Theorem)**
  A graph G is perfect if and only if its complement $\bar{G}$ is perfect

- **Theorem 2 (Strong Perfect Graph Theorem)**
  Perfect graphs are the same as Berge graphs, which are graphs G where neither G nor $\overline{G}$ contain an induced cycle of odd length 5 or more.

Perfect Graphs



- **Perfect Graphs**



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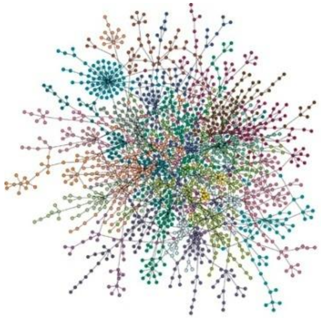
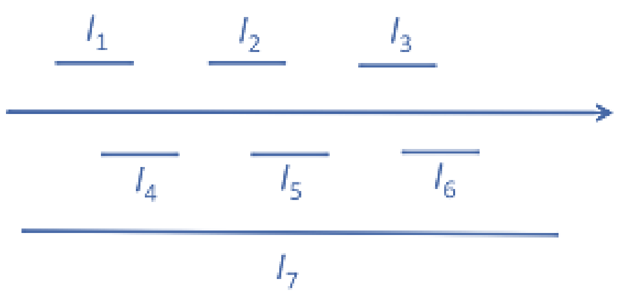


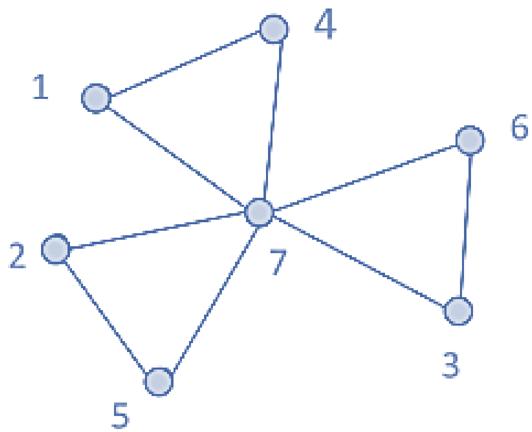
Intersection Graphs

- Let F be a family of nonempty sets. The intersection graph of F is obtained be representing each set in F by a vertex:

$$x \to y \quad \Leftrightarrow S_X \cap S_Y \neq \emptyset$$

- The intersection graph of a family of intervals on a linearly ordered set (like the real line) is called an **Interval graph**.
- An induced subgraph of an interval graph is an interval graph.





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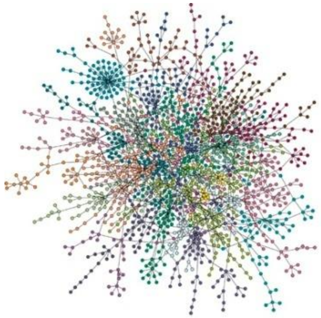

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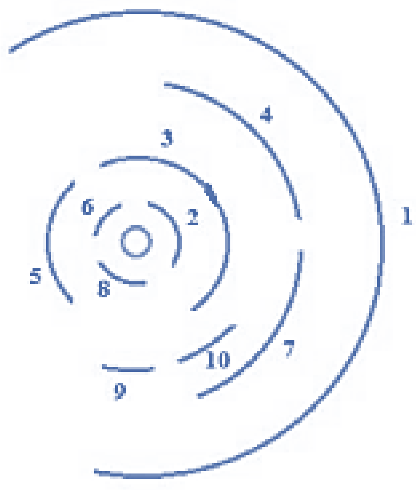
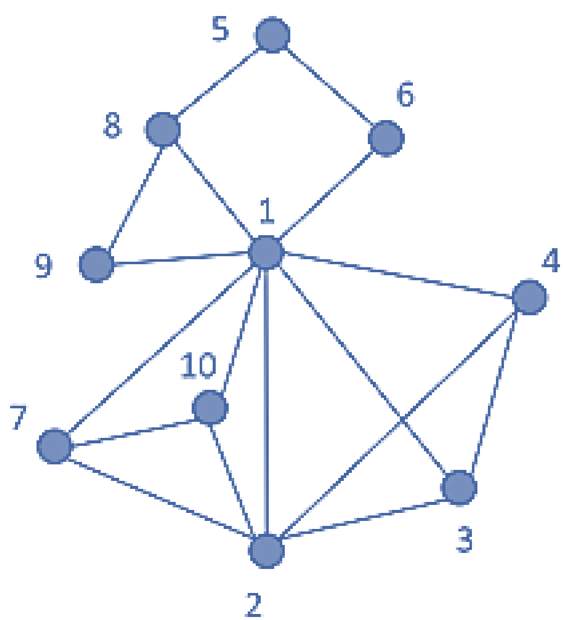
- **Circular-arc graphs** properly contain the internal graphs.





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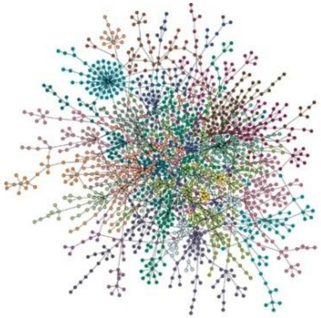

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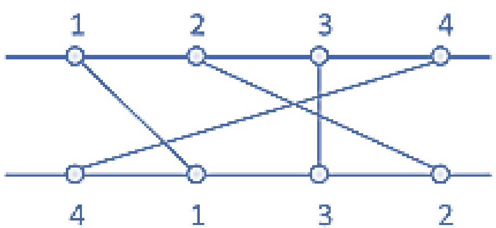
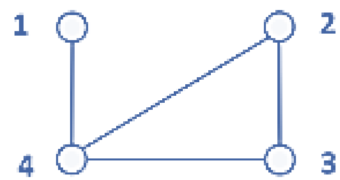
- A **permutation diagram** consists of n points on each of two parallel lines and n straight line segments matching the points.

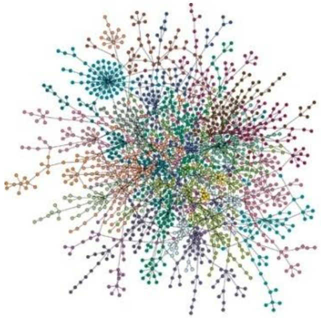


π = [4,1, 3, 2]



G[π]

Perfect Graphs



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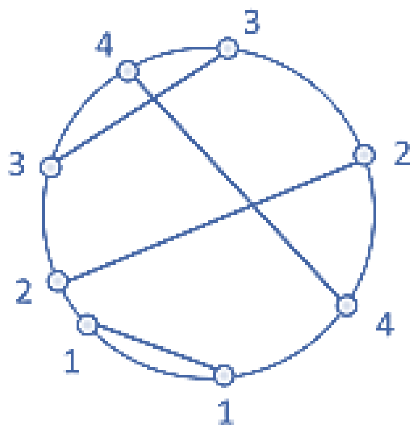
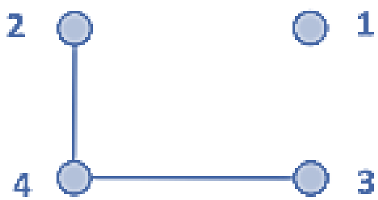
- **Intersecting chords** of a circle





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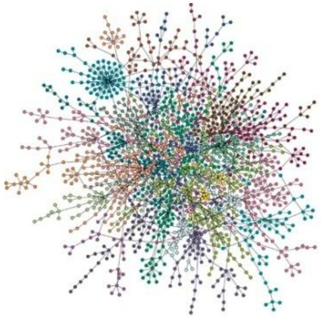
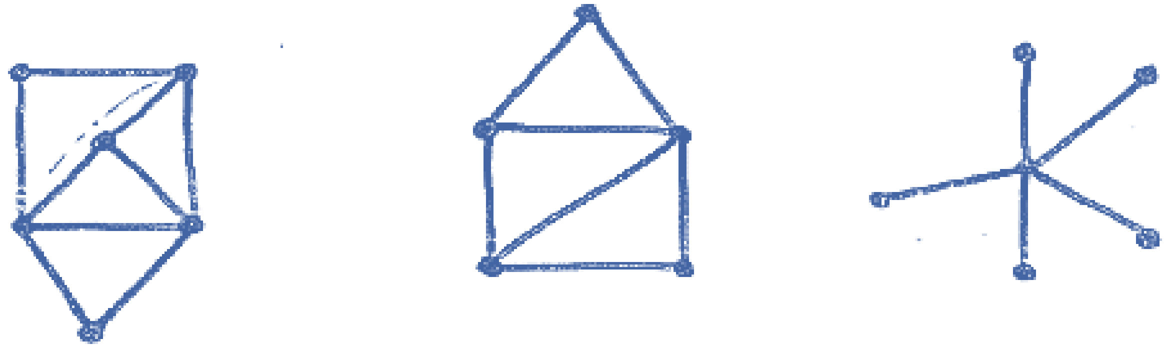


Triangulated Graph Property

- Every simple cycle of length $l > 3$ possesses a chord.

- **Triangulated graphs (or chordal graphs)**



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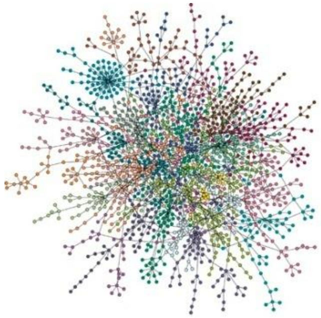
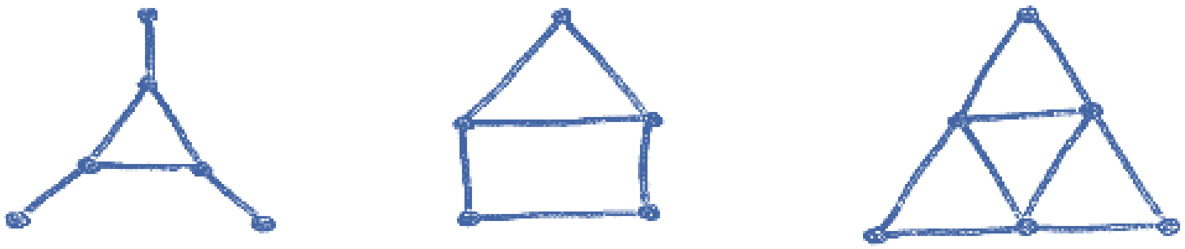


- **Transitive Orientation Property**
  - Each edge can be assigned a one-way direction in such a way that the resulting oriented graph (V, F):

$$ab \in F \text{ and } bc \in F \Rightarrow ac \in F \; (\forall\, a, b, c \in V)$$

  - **Comparability graphs** satisfy the transitive orientation property.



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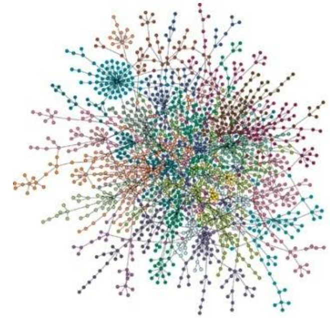

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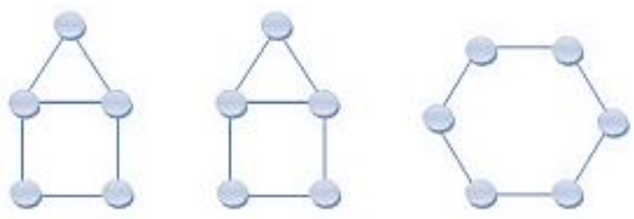

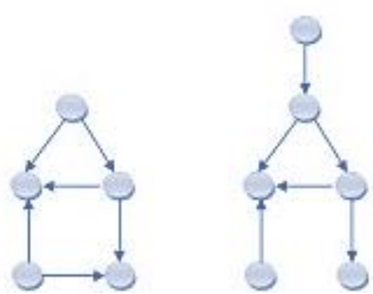

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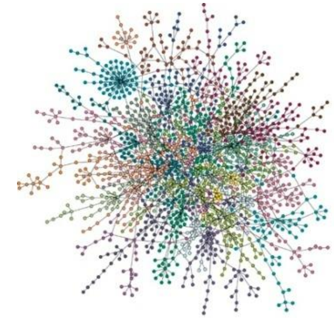
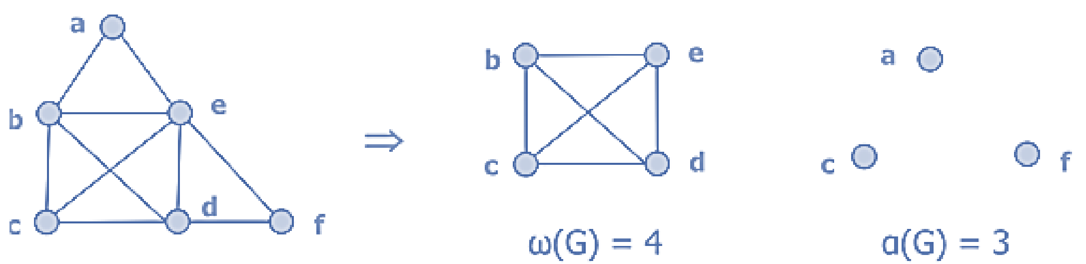


Basic Numbers in Graphs

- **Clique number $\omega(G)$**: the number of vertices in a maximum clique of G
- **Stability number $\alpha(G)$**: the number of vertices in a stable set of max cardinality



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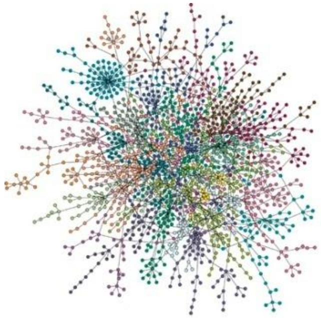
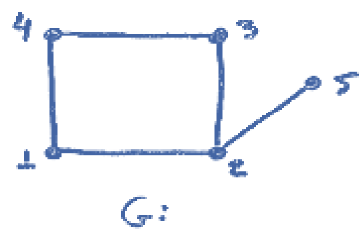


Basic Numbers in Graphs

- **Clique number $\boldsymbol{\omega(G)}$**: the number of vertices in a maximum clique of G
- **Stability number $\boldsymbol{\alpha(G)}$**: the number of vertices in a stable set of max cardinality
- A **clique cover** of size k is a partition $V = C_1 + C_2 + \cdots + C_k$ such that C_i is a clique.
- A **proper coloring** of size c (proper c-coloring) is a partition $V = X_1 + X_2 + \cdots + X_c$ such that X_i is a stable set.
- **Clique cover number $\boldsymbol{\kappa(G)}$** is the size of the smallest possible clique cover of G
- **Chromatic number $\boldsymbol{\chi(G)}$** the smallest possible c for which there exists a proper c-coloring of G.

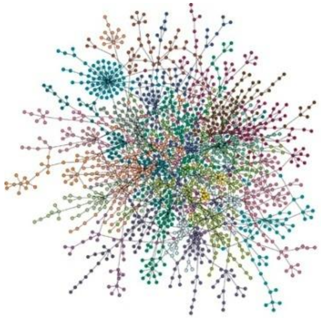


Clique cover $V = \{2,5\} + \{3,4\} + \{1\}$

c-Coloring $V = \{1,3,5\} + \{2,4\}$

$\kappa(G)=3$ $\chi(G)=2$

Perfect Graphs



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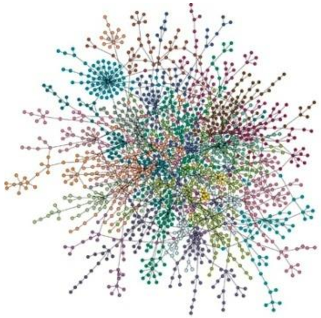
For any graph G it holds that: $\omega(G) \leq \chi(G)$ and $\alpha(G) \leq \kappa(G)$,

while, $\alpha(G) = \omega(\bar{G})$ and $\kappa(G) = \chi(\bar{G})$

Perfect Graphs



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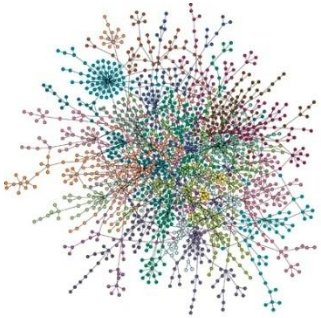
- **χ −Perfect property**: For each induced subgraph G_A of G

$$\chi(G_A) = \omega(G_A)$$

- **α −Perfect property** : For each induced subgraph G_A of G

$$\alpha(G_A) = \kappa(G_A)$$

Perfect Graphs



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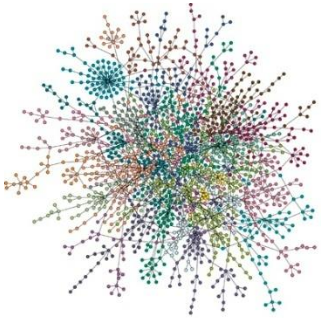
Let $G = (V, E)$ be an undirected graph:

$$(P\,1)\quad \omega(G_A) = \chi(G_A) \quad \forall A \in V$$

$$(P\,2)\quad \alpha(G_A) = \kappa(G_A) \quad \forall A \in V$$

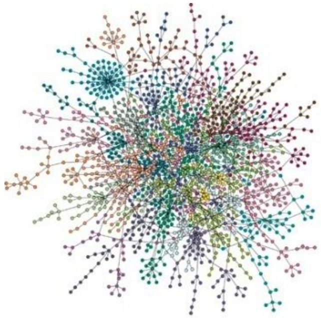
G is called **Perfect**

Perfect Graphs



- **Triangulated Graphs**
  - **Triangulated graphs**, or **Chordal graphs**, or **Perfect Elimination graphs**: G triangulated $\Leftrightarrow$ G has the triangulated graph property (i.e., Every simple cycle of length $l > 3$ possesses a chord)

Perfect Graphs



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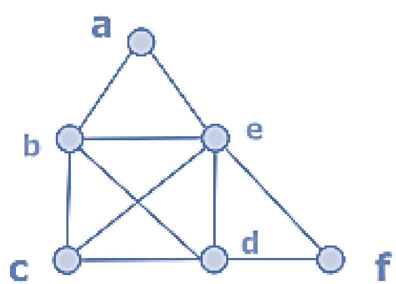
- **Triangulated graphs**, or **Chordal graphs**, or **Perfect Elimination graphs**: G triangulated $\Leftrightarrow$ G has the triangulated graph property (i.e., Every simple cycle of length $l > 3$ possesses a chord)
- **Dirac** showed that: every chordal graph has a simplicial node, a node all of whose neighbors form a clique**.**



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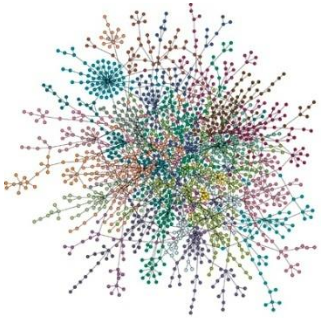

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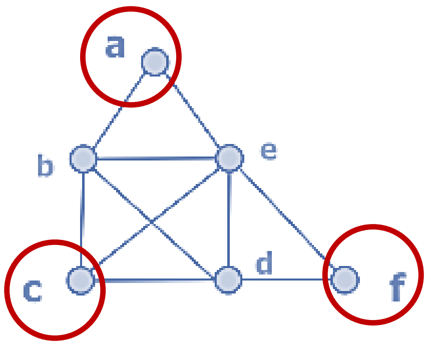
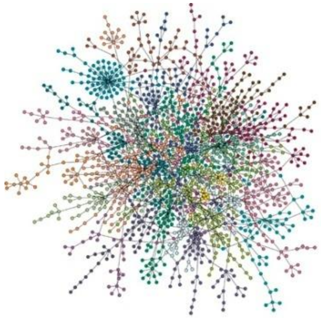


a, c, f	simplicial nodes
b, d, e	non siplicial

Perfect Graphs



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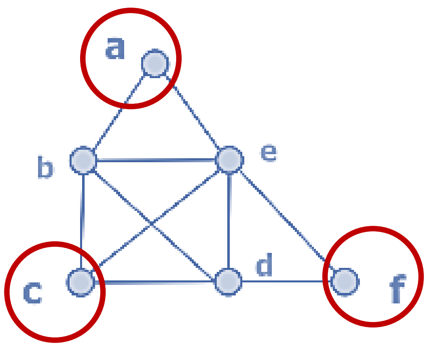
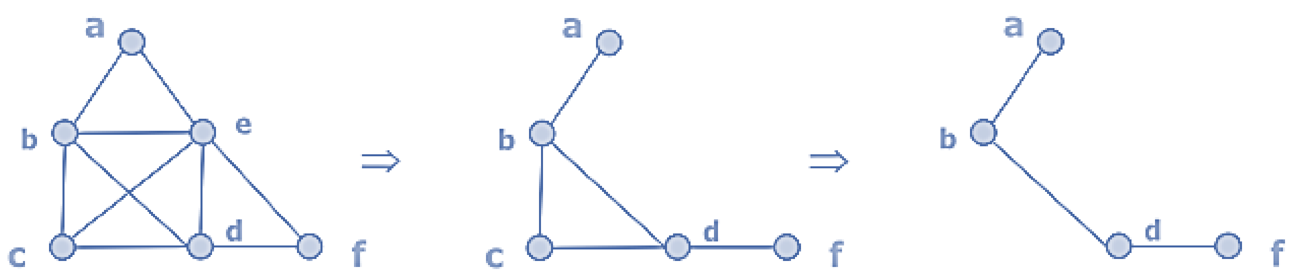


a, c, f	simplicial nodes
b, d, e	non siplicial

- It follows easily from the triangulated property that deleting nodes of a chordal graph yields another chordal graph.



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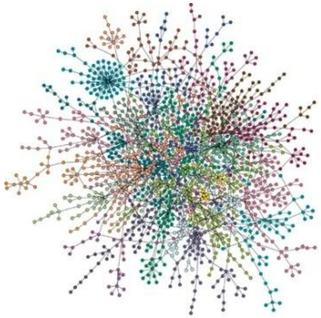

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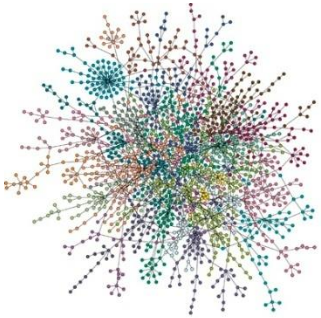
  - **Triangulated graphs**, or **Chordal graphs**, or **Perfect Elimination graphs**: G triangulated $\Leftrightarrow$ G has the triangulated graph property (i.e., Every simple cycle of length $l > 3$ possesses a chord)
  - **Recognition Algorithm :**
    1. **Find** a **simplicial** node of G
    2. **Delete** it from G, resulting G'
    3. **Recourse** on the resulting graph G', until no node remain

PERFECT GRAPHS



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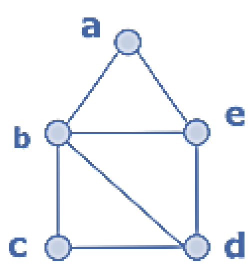
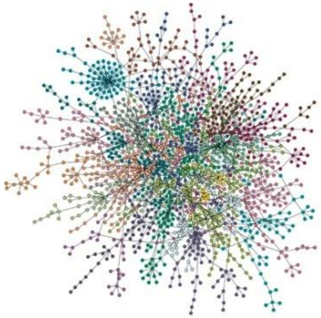
- **Node-Ordering**:
  perfect elimination ordering (PEO), or **perfect elimination scheme.**



(a, c, b, e, d) (c, d, e, a, b) (c, a, b, d, e) ...

Perfect Graphs



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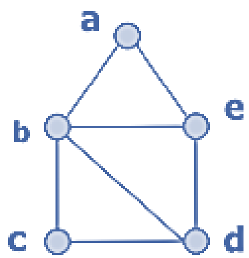
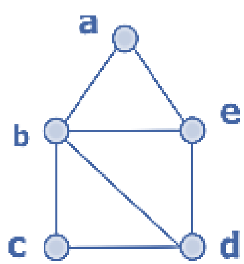
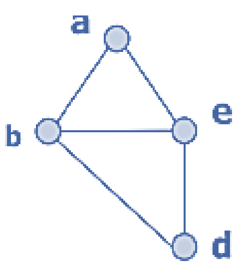


(a, c, b, e, d)   (c, d, e, a, b)   (c, a, b, d, e)   ...

- Let $\sigma = [v_1, v_2, \ldots, v_n]$ be an ordering of the vertices of a graph $G(V, E)$, then σ = **peo** if **each v_i is a simplicial node** to graph $G[\{v_i, v_{i+1}, \ldots, v_n\}]$.



σ = (c, d, e, a, b)



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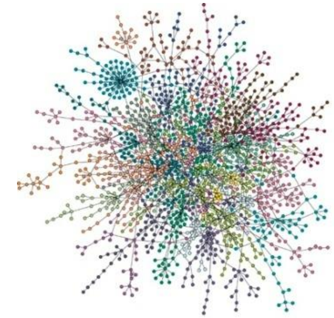

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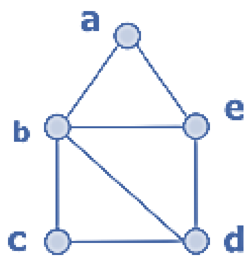
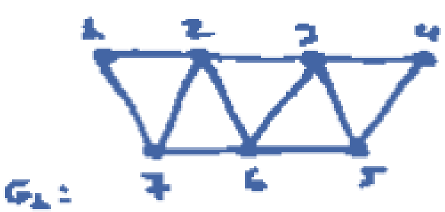
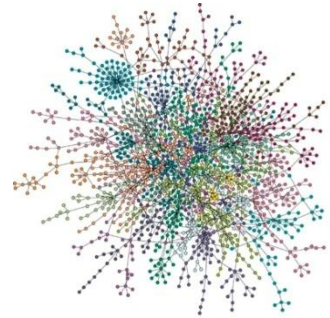


(a, c, b, e, d) (c, d, e, a, b) (c, a, b, d, e) ...

  - Let $\sigma = [v_1, v_2, \dots, v_n]$ be an ordering of the vertices of a graph $G(V, E)$, then σ = **peo** if **each** $\boldsymbol{v_i}$ **is a simplicial node** to graph $G[\{v_i, v_{i+1}, \dots, v_n\}]$.



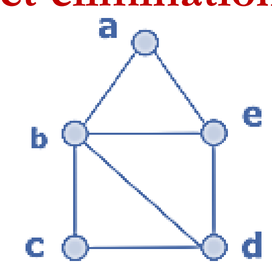
PERFECT GRAPHS



Triangulated Graphs

- **Triangulated graphs**, or **Chordal graphs**, or **Perfect Elimination graphs**: G triangulated $\Leftrightarrow$ G has the triangulated graph property (i.e., Every simple cycle of length $l > 3$ possesses a chord)

- **Node-Ordering**:
  perfect elimination ordering (PEO), or **perfect elimination scheme.**



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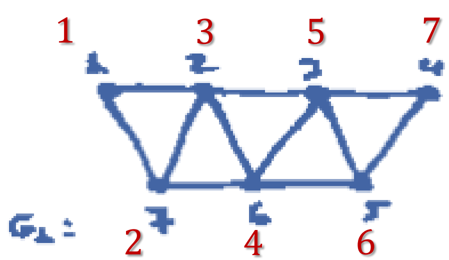
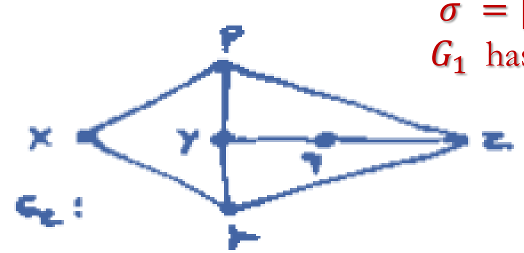
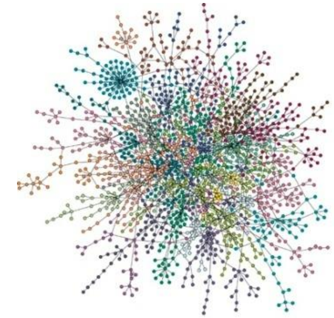




$\sigma = [1, 7, 2, 6, 3, 5, 4]$

G_1 has 96 different peo

Perfect Graphs



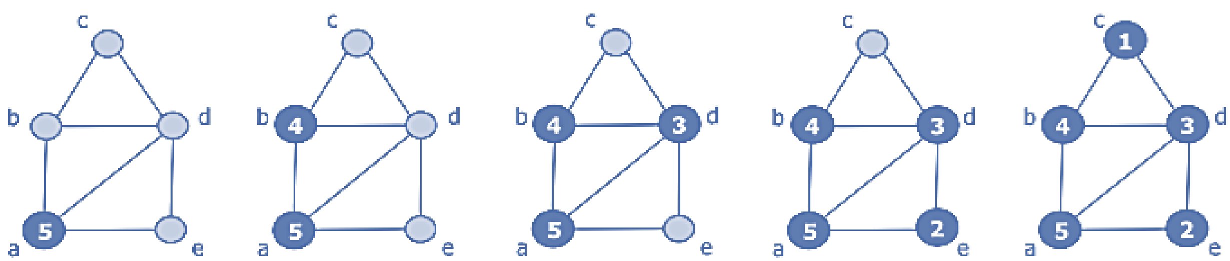
Triangulated Graphs

- **LexBFS Algorithm**

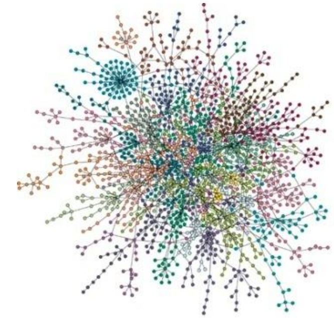
- **Algorithm LexBFS:**

1. for all $v \in V$ do $label(v) := ()$;
2. for $i := |V|$ down to 1 do
   1) select $v \in V$ with lexmax $label\ (v)$;
   2) $\sigma\ (i)\ \leftarrow v$;
   3) for all $u \in\ V\ \cap N(v)$ do
   4) $\quad label\ (u) \leftarrow label\ (u)\ ||\ i$
   5) $V \leftarrow V \setminus \{v\}$;

   end



σ = [a]	σ = [b, a]	σ = [d, b, a]	σ = [e, d, b, a]	σ = [c, e, d, b, a]
L(b) = (4) L(c) = () L(d) = (4) L(e) = (4)	L(c) = (3) L(d) = (43) L(e) = (43)	L(c) = (32) L(e) = (432)	L(c) = (321)	



Perfect Graphs

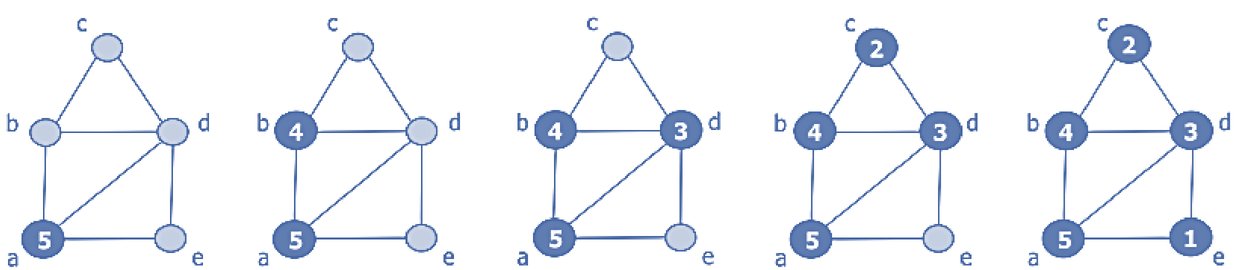
Triangulated Graphs

- **MCS Algorithm**

- **Algorithm MCS:**

1. for $i := |V|$ down to 1 do
   1) select $v \in V$ with max number of numbered neighbors;
   2) number v by i
   3) $\sigma(i) \leftarrow v$;
   4) $V \leftarrow V \setminus \{v\}$;

   end



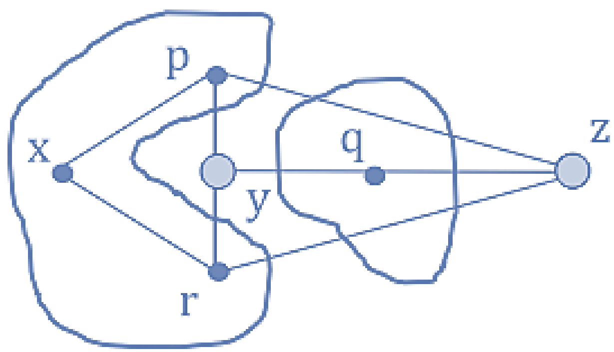
σ = [a] σ = [b, a] σ = [d, b, a] σ = [e, d, b, a] σ = [c, e, d, b, a]

Perfect Graphs

Properties

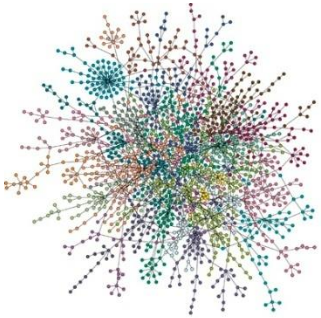
- **Definition:** A subset S of vertices is called a **Vertex Separator** for nonadjacent vertices a, b or, equivalently, $\boldsymbol{a-b}$ **separator,** if in graph G_{V-S} vertices a and b are in different connected components.

- If no proper subset of S in an $a-b$ separator, S is called **Minimal Vertex Separator**.



The set $\{y, z\}$ is a minimal vertex separator for p and q.

PERFECT GRAPHS



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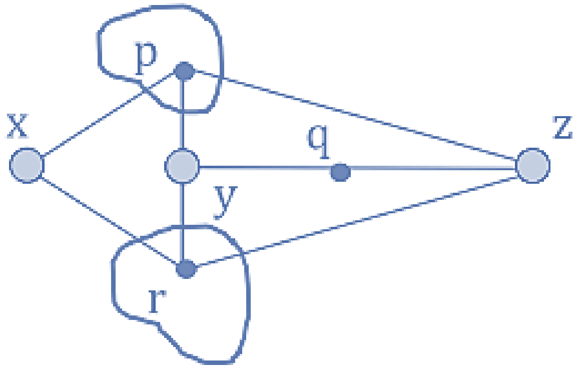
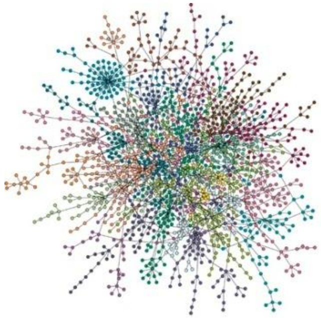


The set $\{x, y, z\}$ is a minimal vertex separator for p and r ($p-r$ separator).

Perfect Graphs



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  (1) G is **triangulated**.
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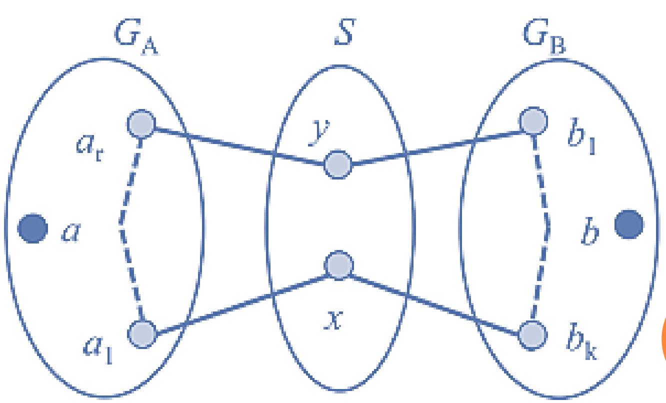
  (3) Every **minimal vertex separator** induces a complete subgraph of G.

  - Proof: (1) ⇒ (3)



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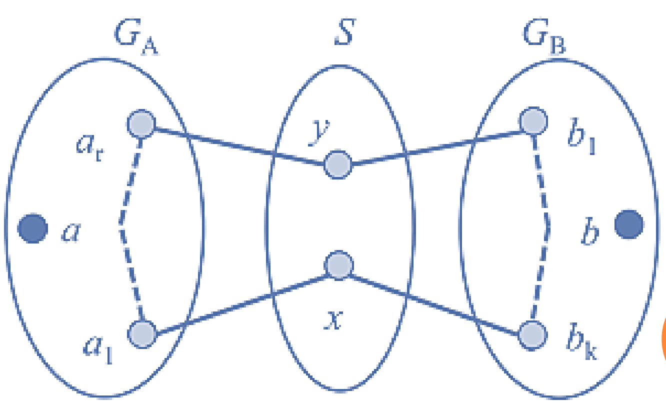
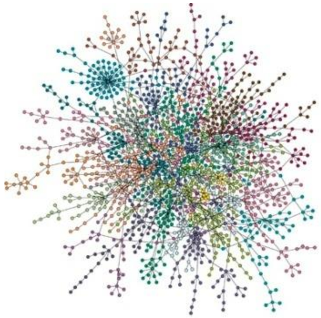
  - Let S be an $a-b$ separator.
  - We will denote G_A, G_B the connected components of G_{V-S} containing a, b.



Perfect Graphs



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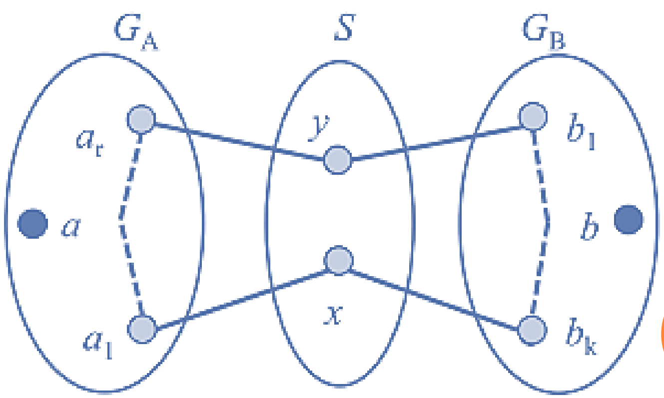
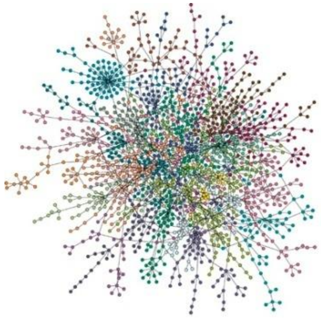
  - Since S is minimal, every vertex $x \in S$ is a neighbor of a vertex in G_A and a vertex in G_B.

  - For any $x, y \in S$, ∃ minimal paths $(x, a_1, \dots, a_i, \dots, a_r, y) a_i \in G_A$ and $(x, b_k, \dots, b_i, \dots, b_1, y) b_i \in G_B$



PERFECT GRAPHS



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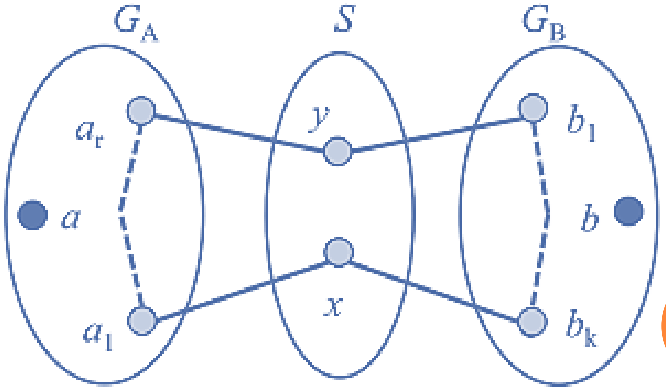
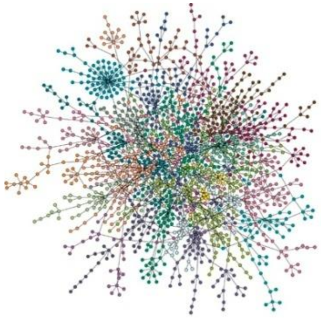
  - Proof: (1) ⇒ (3)

  - Since $[x, a_1, \ldots a_r, y, b_1, \ldots., b_k, x]$ is a simple cycle of length $l \geq 4$, ⇒ ⇒ it contains a chord.



Perfect Graphs



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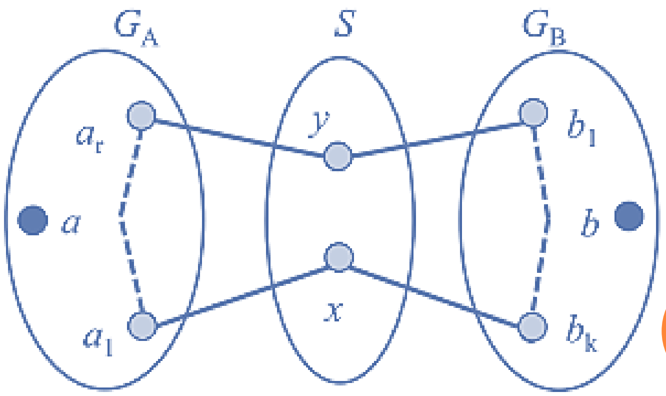
  - Proof: (1) ⇒ (3)

  - For every i, j $a_i b_j \notin E$,
    (S is $a-b$ separator)
    and also $a_i a_j \notin E$, $b_i b_j \notin E$
    (by the minimality of the paths)

  - Thus, $x\ y \notin E$.



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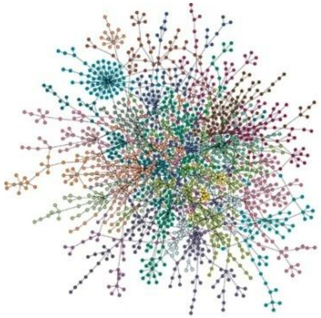

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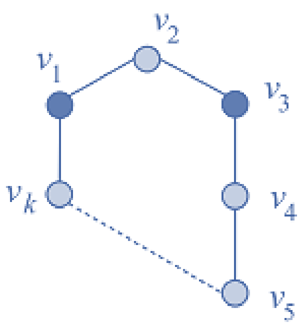
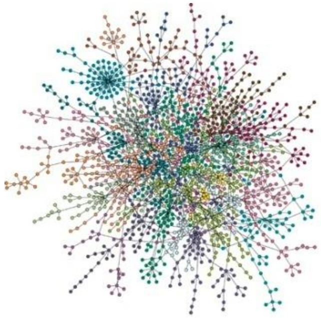
  - Proof: (3) ⇒ (1)



  - Suppose every minimal separator S is a clique
    Let $[v_1, v_2, \ldots, v_k, v_1]$ be a chordless cycle.
  - v_1 and v_3 are nonadjacent.
  - Any minimal $v_1 - v_3$ separator $S_{1,3}$ contains v_2 and at least one of $v_4, v_5, \ldots, v_k$. But vertices v_2, v_i $(i = 4, 5, \ldots, k)$ are nonadjacent ⇒ $S_{1,3}$ does not induce a clique.

Perfect Graphs



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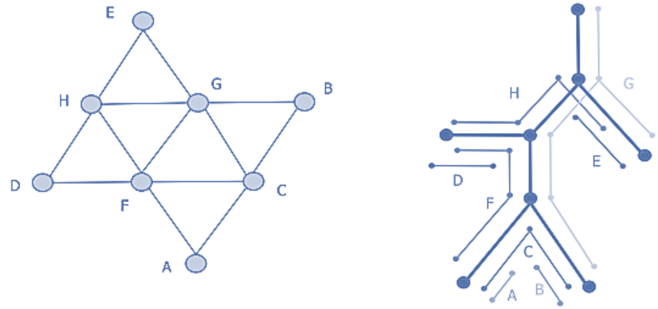
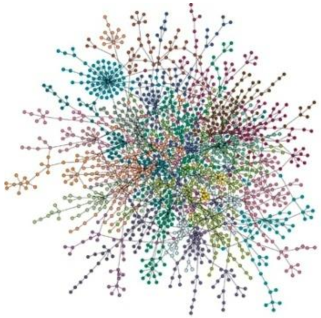
- The chordal graphs are exactly the intersection graphs of subtrees of trees. That is, for a tree T and subtrees $T_1, T_2, \dots, T_n$ of T there is a graph G:
  - its nodes correspond to subtrees $T_1, T_2, \dots, T_n$, and
  - two nodes are adjacent if the corresponding subtrees share a node of T.



Perfect Graphs



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- **Theorem 4:**
  Let G be a graph. The following statements are equivalent.
  (1) G is an interval graph.
  (2) G contains no C_4 and $\bar{G}$ is a comparability graph.
  (3) The maximal cliques of G can be linearly ordered such that, for every vertex x of G the maximal cliques containing vertex x occur consecutively.